up perpendicular to the freestream,

$$w(z) = u(x,y) + \mathrm{I}v(x,y) = (x+\mathrm{I}y)\,[a^2 + (x+\mathrm{I}y^2)]^{-\frac{1}{2}} \quad (8)$$

where $\mathrm{I} \equiv \sqrt{-1}$ and a denotes the half-span of the plate. The lengths Δ_1 and Δ_2 determine the size of that region. The finite difference approximations of the momentum equations are formulated analogously to Eq. (2). The pressure and velocity fields are computed simultaneously by iteration. The relationship between pressure and velocity is given by

$$p_{i,j}^{n+1} = p_{i,j}^{n} - \frac{h^2}{8\Delta t}(\nabla_\delta \cdot v)_{i,j}^{n,m+1} \quad (9)$$

where the equation of continuity is used as a condition of compatibility.[4,5] The divergence of v, $\nabla_\delta \cdot v$, is approximated by centered space difference quotients. This iterative procedure is repeated within each time step, until the velocity field is source free, i.e.,

$$\max |\nabla_\delta \cdot v|_{i,j}^{n,m+1} < h^2$$

The solution is assumed to have obtained an asymptotic steady state, if

$$\max \left| \frac{v_{i,j}^{m+1} - v_{i,j}^{m}}{\Delta t} \right| < \Delta t$$

The computational domain consists of 70×30 grid points in x and y directions, respectively. The time step is $\Delta t = 10^{-3}$ and the constant mesh width is $h = 5 \times 10^{-2}$. The size of the region, where the solution is prescribed, is determined by $\Delta_1 = \Delta_2 = h$.

The streamlines in Fig. 2 show a steady-state, finite difference solution of the Euler equations. This solution seems to be reasonable according to the chosen set of boundary conditions. However, the corresponding lines of constant normalized total pressure displayed in Fig. 2 reveal that the flowfield is not only rotational, but also has no physical meaning at all since a zone exists where the total pressure is higher than the freestream value. These computations were repeated with all parameters chosen as in the first case, but with larger values of Δ_1 and Δ_2. The area with the prescribed analytical solution was increased by setting $\Delta_1 = \Delta_2 = 2h$. Using the same set of boundary conditions and values for the parameters Δt and h, a valid finite difference solution of the Euler equation is obtained. The streamline pattern differs only marginally from the one shown in Fig. 2 and no gradients in total pressure can be detected anywhere.

Conclusions

If an inviscid, incompressible and steady flow around a sharp edge is to be obtained by a finite difference solution of the Euler equations, the region around such a singularity must be treated very carefully. The closer the finite difference solution domain approaches that singular point, the more the leading terms of the truncation error, characterized as diffusion and dispersion of momentum, have a tendency to increase in an unbounded fashion. Then the consistency of the discretized Euler equations with their differential formulation is no longer ensured. Consequently, one cannot judge whether an approximate solution of the Euler equation is obtained or a set of difference equations is solved that has nothing in common with the original differential problem. To avoid these difficulties, it is recommended that the region in the immediate vicinity of a singular sharp edge should be excluded from a finite difference solution of the Euler equations. In that region, the solution should be constructed separately. If available, an analytical solution can be useful, as demonstrated in the numerical tests reported above.

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Stagnation Point Flows with Freestream Turbulence— The Matching Condition

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Introduction

THE calculation of stagnation point flows with imbedded freestream turbulence is of practical importance in many situations. In particular, the combustor exhaust flow impacting on the nozzle guide vanes in a gas turbine is one example, and the heat transfer at a reattachment point of a separated turbulent flow is another. A difficulty has recently been found, however, in a k-ϵ calculation in attaining a proper match between the outer freestream flow and the turbulent/laminar viscous sublayer. Resolution of this problem for the outer freestream flow is the subject of this Note.

Analysis

It is common practice in eddy viscosity approaches to the calculation of turbulent stagnation point flows to assume that at the outer edge of the viscous layer the turbulence quantities such as k (turbulent kinetic energy) correspond to "freestream" values.[1-3] This presents the first dilemma. Far from a body in high Reynolds number turbulence with no mean velocity gradients (no turbulence production), convection and dissipation should balance each other. Sitting on a fluid particle one merely sees a purely decaying turbulence field as the body is approached. The point here is that there is no unique freestream condition and it is changing constantly along a streamline. The resolution of this conflict is, however, fairly easy. A point or plane in the flowfield must be selected as the appropriate "infinity" and the turbulence specified there. In an experiment it could be just downstream of a physical grid. In a gas turbine combustor it could be the last downstream station at which significant turbulence generation takes place on the stagnating streamline of interest. Specifying

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the kinetic energy and dissipation at this point, call them k_0 and ϵ_0, respectively. Now it must be recognized that k and ϵ will change as the stagnation point is approached. The questions are by how much and can a solution be attained all the way to the body.

For the stagnation point configuration of Fig. 1, let a be the nose radius of curvature and u and v are the velocity components in the x and y coordinate directions, respectively. At $y=-\infty$, $\bar{v}=\bar{v}_\infty \approx \bar{v}_0$, depending upon the establishment of the turbulence specification plane. Here overbars indicate conventional time averages. Now consider low-speed constant-density flow. For a high Reynolds number turbulent approach flow, which is assumed herein, there should be no action of turbulent shear stresses upon the mean flow outside a shear layer adjacent to the body. The mean flow will be a potential flow, just as with the high Reynolds number laminar case, outside of the body boundary layer. In this case, the turbulence quantities k and ϵ may be determined by streamline tracking with the primary balance of turbulent kinetic energy between convection, dissipation, and production. Diffusion should be small for the high Reynolds number case, outside the body shear layer. Diffusion requires a little more care, however, and is discussed below.

Let the following nondimensionalizations be made: $U=\bar{u}/\bar{v}_\infty$, $V=\bar{v}/\bar{v}_\infty$, $K=k/k_0$, $E=\epsilon/\epsilon_0$, $I=(k_0/\bar{v}_\infty^2)^{1/2}$, $X=x/a$, $Y=y/a$, $L=(\bar{v}_\infty k_0)/(a\epsilon_0)$. I is recognized as the relative intensity of turbulence at the reference plane, and L has a physical significance to be discussed below. Then, following the k-ϵ formulation of Jones,[4] the following transport equations may be written for K and E along the stagnation streamline for two-dimensional flow, which will be considered here:

$$VK' + E/L - 4C_\mu V'^2 (K^2/E)L = 0 \tag{1}$$

$$VE' + C_2 E^2/KL - 4C_\mu C_1 V'^2 KL = 0 \tag{2}$$

The prime denotes differentiation with respect to Y. The curious factor of 4 in the production terms of Eqs. (1) and (2) arises in this problem from the fact that in the stagnation point problem the stresses that are important are both normal stresses, in contrast to the dominance of the shear stresses in usual boundary-layer treatments. In Eqs. (1) and (2), C_μ, C_2, and C_1 are the turbulent kinematic viscosity and dissipation rate constants conventionally given the values 0.09, 1.9, and 1.45, respectively. If the diffusion terms were retained in Eqs. (1) and (2) terms such as

$$\frac{\partial}{\partial Y}\left(\frac{C_\mu}{\sigma_k}\frac{\partial K}{\partial Y} L I^2\right)$$

would appear in the K equation. It must be demonstrated that diffusion is negligible so that Eqs. (1) and (2) are ordinary differential equations in Y.

Consider now Eq. (1). Far from the body $V' \approx 0$, $V \approx 1$, and the first two terms representing convection and dissipation are in balance. The parameter L clearly represents the ratio of the turbulence decay length normalized by the body nose radius. It will be demanded that this number be of order unity or larger. In this flow regime the $\partial/\partial Y$ is $\mathcal{O}(1/L)$ and since $C_\mu I^2 \ll 1$ usually diffusion is dominated by convection and dissipation. Closer to the body V' can be $\mathcal{O}(1)$ and, if $I^2 \ll 1$, as will be demanded, production dominates diffusion. If the other terms of Eq. (1) can balance the production, as will be shown to be the case, then diffusion may still be neglected. These same arguments apply to Eq. (2). Therefore, it is concluded that these equations may probably be used along the stagnation streamline to determine the K and E evolution. Actually, in analogy with the laminar case, diffusion should not become important until Y enters the shear layer of thickness of order $a/\sqrt{Re_T}$, where Re_T is the turbulence Reynolds number. In this regard, notice that

$$Re_T = \frac{\bar{v}_\infty a}{\nu_T} = \frac{1}{LI^2 C_\mu K^2/E}$$

and recall that this is demanded to be large. Equations (1) and (2) are the proper "outer" inviscid solution equations to be matched ultimately with "inner" viscous solution.

An analytical form for V is often available. For example, for a circular cylinder,[5]

$$V = 1 - [1/(1-Y)^2]$$

In general, however, the form for V is only known analytically very near the stagnation point where

$$V = -\gamma Y, \qquad V' = -\gamma \tag{3}$$

where γ is a constant of order unity. For the cylinder, $\gamma = 2$. Equation (3) will be used below to construct an analytical solution for K and E. For the solution, let $z \equiv K/E$. Equations (1) and (2) then may be combined into

$$z' = \frac{1}{V}(\alpha + \beta z^2 V'^2), \quad \alpha = \frac{C_2 - 1}{L}, \quad \beta = 4C_\mu L(1-C_1) \tag{4}$$

Call point 1 a point at which Eqs. (3) are valid and Eqs. (4) may be integrated from point 1 toward the body to yield

$$\frac{(z-\delta)}{(z_1-\delta)}\frac{(z_1+\delta)}{(z+\delta)} = \left(\frac{Y}{Y_1}\right)^g \tag{5}$$

where

$$\delta^2 = \frac{C_2 - 1}{4C_\mu L^2 (C_1 - 1)\gamma^2}$$

$$g = 4C_\mu^{1/2}(C_2-1)^{1/2}(C_1-1)^{1/2}$$

As $Y \to 0$ and $z \to \delta$, a finite limit; call $\delta = z_s$ the stagnation point limiting value. Then,

$$z_s = \left(\frac{C_2-1}{C_1-1}\right)^{1/2}\frac{1}{2C_\mu^{1/2} L\gamma} \tag{6}$$

Notice that the two end points of z are fixed: $z_0 = 1$ and $z(0) = z_s$ and z_s is a number of $\mathcal{O}(1/L)$.

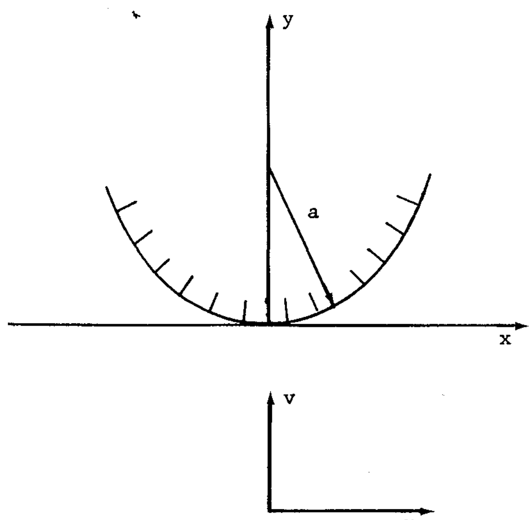

Fig. 1 Schematic of stagnation point flow.

If one examines the solution of Eq. (5), it is found to be one in which the dissipation and production are in balance in the *dissipation* equation. The details are omitted here, but if $C_2 \neq C_1$ this balance does not carry over to the K equation, so that the last two terms are out of balance in Eq. (1). Thus,

$$K \propto a/Y$$

near the body, where a is some nonzero variable. A logarithmic singularity in K is therefore produced which could be matched by no physically correct inner solution near the stagnation point. The conclusion is inescapable: *One or both of the "constants" C_2 and C_1 are incorrect for stagnation flows.* Of the two, C_2 has the most direct support, being selected on the basis of decaying isotropic turbulence. The constant C_1 is usually quoted on the basis of "numerical optimization" by comparison of several calculated flows with experiment.[6] It is the judgment herein that C_1 should be selected equal to C_2 for stagnating flows and that C_2 should retain its conventional value.

It will not be pursued whether or not C_1 could be "fixed" to be a variable dependent upon the local strain rate, for example. Such an endeavor would be warranted in the future. For now, the only statement that will be made is that the conventional vlaue must be changed for stagnating flows. With this change, Eqs. (1) and (2) with Eq. (6) say that as the stagnation point is approached, production and dissipation for both K and E come into balance.

For the case $C_2 = C_1$, a first integral exists for Eqs. (1) and (2):

$$K = z^{1/(1-C_2)} \tag{7}$$

This equation gives an immediate link between the stagnation point condition and the "freestream" condition for

$$K_s = k_s/k_0 = z_s^{1/(1-C_2)} = \left(\frac{1}{2C_\mu^{1/2} L\gamma}\right)^{1/(1-C_2)} \tag{8}$$

Notice in Eq. (8) that since L contains k_0, k_s and k_0 are not simply proportional to each other. In fact,

$$k_s \propto k_0^{C_2/(C_2-1)} \approx k_0^2$$

according to this solution.

Conclusion

For low-speed, two-dimensional, constant-density stagnation point flow with imbedded freestream turbulence, an analytical solution has been found to link the outer flow with an inner viscous layer flow. The physical demand as the stagnation point is approached is that production is in balance with dissipation. However, it is found that, to avoid a singularity, one of the conventional constants in the dissipation rate equation must be mildly changed. The final demonstration is that an analytical relation exists between the stagnation point and freestream "point" turbulence quantities and that the stagnation point values are not simply proportional to the freestream values of k and ϵ.

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The Decay of the Shock Wave from a Supersonic Projectile

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Introduction

It is well known that the sound wave from a supersonic projectile of any shape develops into an N wave after having travelled some distance from the source. This means that the wave profile in time consists of a front shock and a linear time dependence ending with a tail shock. The calculation of the asymptotic decay of these two shocks of equal magnitude appears in the literature, for example, in works by Lighthill,[1] Pierce,[2] and Whitham.[3] It is found that the shock wave amplitude near the Mach cone decreases as $r^{-3/4}$, where r is the radial distance between the flight path and the point of observation. The textbook derivation of this interesting inverse three-quarters power law of asymptotoic decay of cylindrical shock waves is based on nonlinear geometrical acoustics theory with a boundary condition at a distance from the source where linear acoustic theory is still valid.

However, the decay of the shock does not only mean that the shock amplitude decreases, but also that the shock width grows so that the shock finally vanishes. This vanishing of the shock cannot be accounted for by the inviscid wave theory used in Refs. 1-3. In order to obtain the correct wave profile at distances from the source where the shock width has grown so that the shock vanishes, it is necessary to consider dissipative effects, i.e., viscosity and heat conduction. In this Note, instead of using geometrical acoustics, a generalized Burgers equation for cylindrical waves is used. The derivation is similar to that made by the present author in a recent study of nonlinear sound waves from a uniformly moving sinusoidal source.[4] The Mach number of the source is a parameter in the generalized Burgers equation, and it is shown that this equation gives the usual $r^{-3/4}$ decay of the shock wave if the dissipative term is neglected. The development of a cylindrical N wave into a smooth wave profile according to a generalized Burgers equation has been investigated by the present author.[5] The result of this investigation is used herein to give the amplitude and the smooth profile of the sound pulse from a supersonic projectile at distances so far from the source that the shock has broadened and disappeared.

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